Exercise 1. We have

$$\text{v.p.}\frac{1}{x} = (\log|x|)',$$

which shows that for all $\varphi \in \mathscr{S}(\mathbb{R})$,

$$\left|\left\langle \text{v.p.}\frac{1}{x}, \varphi \right\rangle\right| \leq \left|\int_{\mathbb{R}} \log|x|\varphi'(x)dx\right| \leq \left(\int_{-1}^{1} \log\left(\frac{1}{|x|}\right)dx\right)\|\varphi'\|_{\mathrm{L}^\infty(\mathbb{R})}$$
$$+\left(\int_{\mathbb{R}\setminus[-1,1]} \frac{\log|x|}{|x|^2}dx\right)\left\|x^2\varphi'\right\|_{\mathrm{L}^\infty(\mathbb{R})} = 2\left(\|\varphi'\|_{\mathrm{L}^\infty(\mathbb{R})} + \left\|x^2\varphi'\right\|_{\mathrm{L}^\infty(\mathbb{R})}\right).$$

(One need not compute the two integrals above, it suffices to show that they are both finite.)

Exercise 2. We have already seen that $e^x \cdot f \in C^\infty(\mathbb{R})$ in the lecture. Furthermore, for all $n \in \mathbb{N}$, we have

$$0 \leq \limsup_{x\to\infty} x^n f(x) \leq \limsup_{x\to\infty} x^n e^{-x} = 0,$$

which shows that $f \in \mathscr{S}(\mathbb{R})$. We have

$$\langle e^x, f\rangle = \int_0^\infty e^{-\frac{1}{x}}dx = \infty.$$

Therefore, we have $e^x \notin \mathscr{S}'(\mathbb{R})$ whilst $e^x \in \mathscr{D}'(\mathbb{R})$ for the exponential function is locally integrable.

Exercise 3. By Fourier inversion formula, for all $\varphi \in \mathscr{S}(\mathbb{R})$, we have

$$\langle \mathscr{F}(1), \varphi\rangle = \langle 1, \mathscr{F}(\varphi)\rangle = \int_{\mathbb{R}} \widehat{\varphi}(\xi)d\xi = 2\pi\,\varphi(0) = \langle 2\pi\,\delta_0, \varphi\rangle.$$

On the other hand, we have

$$\mathscr{F}(1) = 2\pi\,\delta_0,$$

which makes sense for $\mathscr{F}(\delta_0) = 1$, and we recover the Fourier inversion formula: $\mathscr{F}^2(\delta_0) = 2\pi\delta_0$.

The second computation was made during the lecture and we omit it.

Exercise 4. 1. By the Fourier inversion formula, we have

$$f(x) = \frac{1}{2\pi}\int_{\mathbb{R}^d} \widehat{f}(\xi)e^{ix\cdot\xi}d\xi,$$

and this implies that

$$\|f\|_{\mathrm{L}^\infty(\mathbb{R}^d)} \leq \left\|\widehat{f}\right\|_{\mathrm{L}^1(\mathbb{R}^d)} \leq \left(\int_{\mathbb{R}^d} \frac{dx}{(1+|\xi|^2)^s}\right)^{\frac{1}{2}}\|f\|_{\mathrm{H}^s(\mathbb{R}^d)} = \left(\beta(d)\int_0^d \frac{r^{d-1}dr}{(1+r^2)^s}\right)^{\frac{1}{2}}\|f\|_{\mathrm{H}^s(\mathbb{R}^d)}.$$

When $r \to \infty$, we have

$$\frac{r^{d-1}}{(1+r^2)^s} \simeq \frac{1}{r^{1+2s-d}},$$

and since $2s - d > 0$, the integral converges and the inequality is proven.

2. If $u,v\in\mathscr{S}(\mathbb{R}^d)$, we have
$$\widehat{uv}(\xi)=\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\widehat{u}(\xi-\eta)\widehat{v}(\eta)d\eta.$$
Therefore, writing for simplicity $\langle x\rangle^s=\left(1+|x|^2\right)^{\frac{s}{2}}$, we get
$$\langle xi\rangle^s|\widehat{uv}(\xi)|\leq\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}\left(\langle\xi-\eta\rangle^s\widehat{u}(\xi-\eta)\right)\widehat{v}(\eta)d\eta+\frac{1}{(2\pi)^d}\int_{\mathbb{R}^d}|\widehat{u}(\xi-\eta)|\langle\eta\rangle^s|\widehat{v}(\eta)|d\eta.$$
Since $\widehat{u},\widehat{v}\in L^1(\mathbb{R}^d)$, we can apply Young's inequality $L^1*L^2\subset L^2$ to conclude the proof.

Exercise 5. 1. Indeed, we have
$$\begin{aligned}p\int_0^\infty t^{p-1}\mu\left(X\cap\{x:|f(x)|>t\}\right)dt&=p\int_0^\infty t^{p-1}\left(\int_X\mathbf{1}_{\{|f(x)|>t\}}d\mu(x)\right)dt\\&=\int_X\left(p\int_0^\infty\mathbf{1}_{\{|f(x)|>t\}}dt\right)d\mu(x)=\int_X\left(p\int_0^{|f(x)|}t^{p-1}dt\right)d\mu(x)=\int_X|f(x)|^pd\mu(x).\end{aligned}$$

2. Indeed, we see at once that
$$\{|f|>t\}\subset\left\{|g|>\frac{t}{2}\right\}\cup\left\{|h|>\frac{t}{2}\right\},$$
and the result follows by additivity of the integral. If $x\in\{|f|>t\}$, then $|f(x)|>t$, and since $|f(x)|=|g(x)+h(x)|$, we must have $|g(x)|>\dfrac{t}{2}$ or $|h(x)|>\dfrac{t}{2}$ (otherwise, we get $|f(x)|\leq|g(x)|+|h(x)|\leq t$, a contradiction).

3. Since $\widehat{g_t}+\widehat{h_t}=\widehat{f}$, by unicity of the Fourier transform, we deduce that $f=g_t+h_t$. If $\left\{|g_t|>\dfrac{t}{2}\right\}=\varnothing$, the previous two questions show that
$$\begin{aligned}\int_{\mathbb{R}^d}|f(x)|^pdx&\leq p\int_0^\infty t^{p-1}\mathscr{L}^d\left(\left\{|h_t|>\frac{t}{2}\right\}\right)dt\leq p\int_0^\infty t^{p-1}\frac{4}{t^2}\left(\int_{\mathbb{R}^d}|h_t(x)|^2dx\right)dt\\&=4p\int_0^\infty t^{p-3}\left\|h_t\right\|_{\mathrm{L}^2(\mathbb{R}^d)}dt,\end{aligned}$$
where we used Markov's inequality
$$\mathscr{L}^d(\{|\varphi|>A\})\leq\frac{1}{A^2}\int_{\mathbb{R}^d}|\varphi(x)|^2d\mathscr{L}^d(x).$$

4. We have by Cauchy-Schwarz inequality
$$|g_t(x)|\leq\frac{1}{(2\pi)^d}\int_{B(0,A_t)}|\widehat{f}(\xi)|d\xi\leq\frac{1}{(2\pi)^d}\left(\int_{B(0,A_t)}|\xi|^{-2s}d\xi\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^d}|\xi|^{2s}|\widehat{f}(\xi)|^2d\xi\right)^{\frac{1}{2}}.$$
Then, we have
$$\int_{B(0,A_t)}|\xi|^{-2s}d\xi=\beta(d)\int_0^{A_t}r^{d-1-2s}dr=\frac{\beta(d)}{d-2s}A_t^{d-2s}.$$
Therefore, we define A_t such that
$$\frac{1}{(2\pi)^d}\sqrt{\frac{\beta(d)}{d-2s}}A_t^{\frac{d}{2}-s}=\frac{t}{2}$$
and this yields $\|g_t\|_{\mathrm{L}^\infty(\mathbb{R}^d)}\leq\dfrac{t}{2}$, and as g_t is a continuous function, this implies that $\left\{|g_t|>\dfrac{t}{2}\right\}=\varnothing$. For simplicity, write from now on $A_t=C(d,s)t^{\frac{d}{2}-s}$.

5. We finally get the inequality

$$\begin{aligned}\int_{\mathbb{R}^d}|f(x)|^p dx &\leq 4p(2\pi)^d\int_0^{\infty}\left(\int_{\mathbb{R}^d\setminus\overline{B}(0,A_t)} t^{p-3}|\widehat{f}(\xi)|^2 d\xi\right)dx\\ &= 4p(2\pi)^d\int_{\mathbb{R}^d}|\widehat{f}(\xi)|^2\left(\int_0^{C(d,s)|\xi|^{\frac{d}{2}-s}} t^{p-3}dt\right)d\xi\\ &= \frac{4p(2\pi)^d}{p-2}(C(d,s))^{p-2}\int_{\mathbb{R}^d}|\xi|^{\left(\frac{d}{2}-s\right)(p-2)}|\widehat{f}(\xi)|^2 d\xi\\ &= \frac{4p(2\pi)^d}{p-2}(C(d,s))^{p-2}\int_{\mathbb{R}^d}|\xi|^{\frac{d(p-2)}{p}}|\widehat{f}(\xi)|^2 d\xi,\end{aligned}$$

where we used that $\dfrac{d}{2}-s=\dfrac{d}{p}$.